

MODULI SPACES OF STABLE POLYGONS AND SYMPLECTIC STRUCTURES ON $\overline{\mathcal{M}}_{0,n}$

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ABSTRACT. In this paper, certain natural and elementary polygonal objects in Euclidean space, *the stable polygons*, are introduced, and the novel moduli spaces $\mathfrak{M}_{r,\varepsilon}$ of stable polygons are constructed as complex analytic spaces. Quite unexpectedly, these new moduli spaces are shown to be projective and isomorphic to the moduli space $\overline{\mathcal{M}}_{0,n}$ of the Deligne-Mumford stable curves of genus 0. Further, built into the structures of stable polygons are some natural data leading toward to a family of (classes of) symplectic (Kähler) forms. To some degree, $\mathfrak{M}_{r,\varepsilon}$ may be considered as symplectic counterparts of $\overline{\mathcal{M}}_{0,n}$ and Kapranov's Chow quotient construction of $\overline{\mathcal{M}}_{0,n}$. All these together brings up a new tool to study the Kähler topology of $\overline{\mathcal{M}}_{0,n}$.

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1. INTRODUCTION

The moduli spaces of Riemann surfaces were introduced by Riemann in the nineteenth century and have been since great sources of interest and study. Most recently, they have evolved as essential ingredients in symplectic topology and mirror symmetry. In these theories, a guiding principle is that the geometry of an almost complex manifold may be obtained by studying the space of all (pseudo-) holomorphic curves on

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the manifold. One instance is the Gromov-Witten invariants. A key building block in that theory is the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$ of pointed Riemann surfaces of genus g . Symplectic topology and mirror symmetry are, of course, not the only places where $\overline{\mathcal{M}}_{g,n}$ is important. These moduli spaces are also rich subjects of study in Thurston's hyperbolic geometry and Teichmüller theory (and so on). On yet another important aspect of geometry, the genus zero case $\overline{\mathcal{M}}_{0,n}$ was recently effectively used by Kawamata to prove a higher codimensional adjunction formula in algebraic geometry.

The basic questions we will address in this paper are: *how do we construct these compactified moduli spaces and furthermore what geometric information can we derive from the construction?*

Our answers to these questions are very simple and rather unexpected.

We will only look at the special case of genus zero, namely, the moduli space $\overline{\mathcal{M}}_{0,n}$ of stable n -pointed Riemann spheres. Set-theoretically, a generic point in $\overline{\mathcal{M}}_{0,n}$ is an equivalence class of Riemann spheres with n distinct ordered points. All other points on the boundary of $\overline{\mathcal{M}}_{0,n}$ are to be obtained by abiding to the following principle: whenever some of marked points on a sphere come together, we attach another sphere, called a bubble, at the coinciding position and let the coinciding points get separated on the bubble sphere. This process should go on until all points get separated (see the picture below).

$$\overline{\mathcal{M}}_{0,n} =$$

Figure 1

In this paper, We will (quite unexpectedly) build $\overline{\mathcal{M}}_{0,n}$ using totally differently materials. In short, we shall give *symplectic* constructions of $\overline{\mathcal{M}}_{0,n}$ by using only elementary geometric combinatorial objects in the Euclidean space which we will name as *stable polygons*. This

yields some quite unexpected results which among other things include a family of naturally and automatically built-in (classes of) symplectic Kähler forms on $\overline{\mathcal{M}}_{0,n}$. We believe that this will eventually allow us to quantitatively determine (in a forthcoming paper) the Kähler cone and dually the Mori cone of effective curves on $\overline{\mathcal{M}}_{0,n}$ (cf. §§8 and 9).

Our approach is based upon a beautiful connection between symplectic and algebraic geometry, discovered in late 70's and early 80's. Briefly, let (X, ω) be a symplectic manifold that underlies a polarized projective manifold (X, \mathcal{L}) , where ω is a symplectic form on X and \mathcal{L} is a positive line bundle such that

$$c_1(\mathcal{L}) = [\omega] \in H^2(X, \mathbb{Z})$$

Suppose a compact Lie group K acts symplectically on X with its complexification $G = K_{\mathbb{C}}$ acting holomorphically. Then in a canonical way, we have a correspondence between quotients in symplectic category and quotients in projective algebraic category. More precisely, assume that K acts on X in a Hamiltonian fashion, that is, there is an equivariant moment map (depending on ω)

$$\Phi_{\omega} : X \rightarrow \mathfrak{k}^*$$

where \mathfrak{k} is the Lie algebra of K . Then the orbit space $\Phi_{\omega}^{-1}(0)/K$ carries a naturally induced symplectic structure away from singularities. This is a symplectic quotient. On the other hand, there is a dense open subset

$$X^{ss}(\mathcal{L}) \subset X,$$

the set of all semistable points determined by \mathcal{L} . By Mumford's geometric invariant theory, $X^{ss}(\mathcal{L})$ has a quotient $X^{ss}(\mathcal{L})//G$ as a projective variety. The beautiful correspondence mentioned above is that the symplectic quotient $\Phi_{\omega}^{-1}(0)/K$ is, in a canonical way, homeomorphic to the projective quotient $X^{ss}(\mathcal{L})//G$ (Mumford, Atiyah, Guillemin-Sternberg, Kempf-Ness, Ness, Kirwan, and others).

We now describe *stable polygons* in \mathbb{R}^3 , the building blocks of our construction of $\overline{\mathcal{M}}_{0,n}$. A polygon in \mathbb{R}^3 is a collection of ordered vectors (edges) $\{e_1, \dots, e_n\}$ that add up to zero,

$$e_1 + \dots + e_n = 0.$$

Two polygons are equivalent if one can be obtained from the other by a rigid Euclidean motion. Consider any n -gon with side lengths, $\mathbf{r} = (r_1, \dots, r_n)$, then the deformation space of this n -gon modulo equivalence is

$$M_{\mathbf{r}} = \{(e_1, \dots, e_n) | e_1 + \dots + e_n = 0, |e_i| = r_i, 1 \leq i \leq n\} / \sim.$$

This is a symplectic space with a symplectic form $\omega_{\mathbf{r}}$ away from some (possibly) isolated singularities (for more about $M_{\mathbf{r}}$, see, e.g., [13]. For the ring structure on $H^*(M_{\mathbf{r}})$, see [10]).

To introduce stable polygons, we fix a term once and for all in this paper: edges are said to be *parallel* if they point the same direction; edges pointing in opposite directions are considered *anti-parallel*, not parallel.

A generic polygon, that is, a polygon with no parallel edges, is stable in our sense. All other stable polygons are to be obtained by abiding the following principle (much of which is in the spirit of Fulton-MacPherson [4]): whenever a set of edges of a polygon becomes parallel, we introduce an (independent) generic polygon, called a bubble, whose edges inherit the lengths of the abovementioned parallel edges except the longest one. The longest length is the sum of the lengths of the parallel edges minus a carefully chosen small positive number ϵ . (This ϵ has a precise quantity control and carries significant symplecto-geometric meaning, consult Theorems 1.2 and 1.3 below.) This process should go on until all sets of parallel edges are properly addressed.

In short, a stable polygon is a collection of labeled (but not ordered) polygons that grows out of an ordinary polygon by introducing “bubble” polygons. The moduli space of all such stable polygons is denoted by $\mathfrak{M}_{\mathbf{r},\epsilon}$. See the illustration below.

$$\mathfrak{M}_{\mathbf{r},\epsilon} =$$

Figure 2

One should note that the moduli space $\mathfrak{M}_{\mathbf{r},\epsilon}$ of stable polygons (in a very interesting manner) depends on two parameters: a fixed length vector \mathbf{r} and a collection $\epsilon = \{\epsilon\}$ of carefully chosen small positive

numbers. These built-in data in stable polygons encode significant symplecto-geometric information for $\overline{\mathcal{M}}_{0,n}$.

Our main theorems are

Theorem 1.1. (§§5 and 7) *The moduli space $\mathfrak{M}_{\mathbf{r},\varepsilon}$ has a natural structure of a compact complex manifold and is biholomorphic to the complex manifold that underlies the projective variety $\overline{\mathcal{M}}_{0,n}$.*

Theorem 1.2. (§6) *By forgetting all the bubbles, we obtain a natural projection*

$$\pi_{\mathbf{r},\varepsilon} : \mathfrak{M}_{\mathbf{r},\varepsilon} \rightarrow M_{\mathbf{r}}$$

that is holomorphic and bimeromorphic. It is the iterated blowup of $M_{\mathbf{r}}$ along (the proper transforms of) some explicitly described smooth subvarieties Y_{α} when $M_{\mathbf{r}}$ is smooth. When $M_{\mathbf{r}}$ is singular, $\pi_{\mathbf{r},\varepsilon} : \mathfrak{M}_{\mathbf{r},\varepsilon} \rightarrow M_{\mathbf{r}}$ is the composite of a canonical resolution of singularities followed by explicit iterated blowups.

In fact, more may be true. With the aid of the so-called symplectic ε -blowups (e.g., see [20]), $\mathfrak{M}_{\mathbf{r},\varepsilon}$ (hence $\overline{\mathcal{M}}_{0,n}$) comes equipped with a symplectic form $\Omega_{\mathbf{r},\varepsilon}$ whose cohomology class is uniquely defined so that the map

$$\pi_{\mathbf{r},\varepsilon} : (\mathfrak{M}_{\mathbf{r},\varepsilon}, \Omega_{\mathbf{r},\varepsilon}) \rightarrow (M_{\mathbf{r}}, \omega_{\mathbf{r}})$$

can be interpreted as a symplectic blowup. The ambiguity of the so-called symplectic blowups provides a natural and interesting viewpoint to see how natural it is the choices of ε in the definition of stable polygons. This point calls for further investigation.

To close this introduction, some remarks are in order. Firstly, $\overline{\mathcal{M}}_{0,n}$ has a canonical Kähler class, the Weil-Peterson class. We believe that this form should be realized by some choice of $(\mathbf{r}, \varepsilon)$. It has to come from regular n -gons. The choice of ε is harder to guess (consult 4.6). Secondly, by above, the collection of all (legal¹) $(\mathbf{r}, \varepsilon)$ should shed a light on the Kähler cone of $\overline{\mathcal{M}}_{0,n}$ (a conjecture is formulated in §9). Finally, Thurston made remarks at author's seminar talk at UC, Davis that the same strategy can be applied to unordered n -pointed Riemann spheres. Then the role of polygons will then be replaced by collections of unordered vectors that add up to zero. He also pointed out to the author that polyhedra may also be used to study $\overline{\mathcal{M}}_{0,n}$ ([23]). Fried at author's talk at UC, Irvine, also made similar suggestions about unordered n -pointed Riemann spheres.

Here is the structure of this paper. Sections 2 provides necessary backgrounds on the symplectic geometry of the ordinary n -gons. In

¹See 2.1 and 4.6 for the legal ranges of \mathbf{r} and ε .

Section 3, *favorable* moduli spaces of ordinary n -gons are characterized in terms of their defining side length vectors (chambers). They are useful throughout the paper. Section 4 introduces our principal objects of study, stable polygons, mainly in set-theoretic terms. In Section 5, we show how naturally stable polygons converge together to form a smooth compact complex analytic variety. Relation with the moduli spaces of the ordinary polygons follows readily: section 6 decomposes $\mathfrak{M}_{r,\varepsilon}$ as explicit iterated blowups of M_r . Section 7 connects our new moduli spaces to the moduli space $\overline{\mathcal{M}}_{0,n}$ of the Deligne-Mumford stable n -pointed curves of genus zero. This leads to a new angle to look at the Kähler topology of $\overline{\mathcal{M}}_{0,n}$ which is discussed in the end.

There are some previous works on the moduli spaces of *ordinary* polygons ([24], [18], [13], [9], among others) which may fall into the general framework on relations between geometric invariant theory quotients and symplectic reductions ([21], [7], [17], [1], [11], [2], among others). The idea that $\overline{\mathcal{M}}_{0,n}$ ought to have a counterpart in terms of symplectic geometry was based on the observation ([14] and [11]) on a natural connection between GIT quotients and Chow (Hilbert) quotient.

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2. GEOMETRY OF ORDINARY POLYGONS

Many of the results on ordinary polygons that we shall collect below follows directly from some well-known general theory. See [13], for example, for a nice self-contained treatment.

2.1. A (spatial) polygon with n sides in Euclidean space is determined by its labeled n vertices $\{v_1, \dots, v_n\}$ and these vertices are joined in cyclic order by the directed edges

$$\{e_1, \dots, e_n\}$$

where e_i starts from v_i and ends at v_{i+1} (here we set $v_{n+1} = v_1$). Since the n -gon is closed, $\{e_1, \dots, e_n\}$, regarded as vectors, add up to zero,

$$e_1 + \dots + e_n = 0.$$

We consider the following equivalence relation among polygons. Two n -gons are equivalent if one can be obtained from the other by the action of an orientation preserving Euclidean isometry.

Figure 3 illustrates three 12-gons. (The last two will be used later for other illustrations.)

Figure 3

Throughout this paper, $M_{\mathbf{r}}$ will stand for the moduli space² of n -gons with the side length vector $\mathbf{r} = (r_1, \dots, r_n)$. The normalized side length vector of \mathbf{r} , denoted by $\bar{\mathbf{r}}$, is defined to be $\frac{2}{L(\mathbf{r})}\mathbf{r}$ where

$$L(\mathbf{r}) = r_1 + \dots + r_n$$

is the perimeter of the polygons of $M_{\mathbf{r}}$. As is well-known, $\bar{\mathbf{r}}$ lies in the hypersimplex

$$\mathbb{D}_2^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1, i = 1, \dots, n, \sum_i x_i = 2\}.$$

To see the fine structures on $M_{\mathbf{r}}$, we will appeal to the theory of symplectic reductions and Kähler geometric invariant theory (GIT), from which one will see that $M_{\mathbf{r}}$ and $M_{\bar{\mathbf{r}}}$ are biholomorphic as complex analytic spaces and their symplectic (Kähler) forms (away from singularities) are proportional by the scalar $\frac{2}{L}$.

²The topology of this space is rather easy to see. We will review some of its finer structures soon.

2.2. Consider the diagonal action of the group $\mathrm{PGL}(2, \mathbb{C})$ of projective transformations on $(\mathbb{P}^1)^n$ and identify its maximal compact subgroup with $\mathrm{SO}(3)$. Identify \mathbb{P}^1 with the unit sphere S^2 in \mathbb{R}^3 , $\mathrm{SO}(3)$ -equivariantly. Let $\mathrm{Vol}(S^2)$ be the volume form on S^2 . Given any \mathbf{r} with $r_1, \dots, r_n > 0$,

$$\mathcal{L}_{\mathbf{r}} = r_1 p_1^* \mathrm{Vol}(S^2) + \dots + r_n p_n^* \mathrm{Vol}(S^2)$$

is a Kähler symplectic form on $(\mathbb{P}^1)^n$ with respect to which the group $\mathrm{SO}(3)$ acts on $(\mathbb{P}^1)^n$ in a Hamiltonian fashion. Here p_i ($1 \leq i \leq n$) is the projection from $(\mathbb{P}^1)^n$ onto the i -th factor. The moment map

$$\Phi_{\mathbf{r}} : (\mathbb{P}^1)^n \rightarrow \mathfrak{so}(3)^* \cong \mathbb{R}^3$$

determined by this Hamiltonian action is given as:

$$\Phi_{\mathbf{r}}(x_1, \dots, x_n) = r_1 x_1 + \dots + r_n x_n.$$

The correspondence between symplectic reductions and Kähler (GIT, in the case that \mathbf{r} is rational) quotients asserts that

$$\Phi_{\mathbf{r}}^{-1}(0)/\mathrm{SO}(3) \cong (\mathbb{P}^1)_{ss}^n(\mathbf{r})//\mathrm{PGL}(2, \mathbb{C})$$

which is induced by the inclusion

$$\Phi_{\mathbf{r}}^{-1}(0) \subset (\mathbb{P}^1)_{ss}^n(\mathbf{r}),$$

where $(\mathbb{P}^1)_{ss}^n(\mathbf{r})$ denotes the set of $\mathrm{PGL}(2, \mathbb{C})$ -semistable points with respect to the Kähler form $\mathcal{L}_{\mathbf{r}}$. We point out that $\Phi_{\mathbf{r}}^{-1}(0) \neq \emptyset$ (or equivalently, $(\mathbb{P}^1)_{ss}^n(\mathbf{r}) \neq \emptyset$) if and only if $\bar{\mathbf{r}} \in \mathbb{D}_2^n$. In this case, the quotient space $\Phi_{\mathbf{r}}^{-1}(0)/\mathrm{SO}(3) \cong (\mathbb{P}^1)_{ss}^n(\mathbf{r})//\mathrm{PGL}(2, \mathbb{C})$ has the expected dimension $n - 3$ if and only if $\bar{\mathbf{r}} \in \mathrm{int}\mathbb{D}_2^n$ (see [11]). For this reason, we make the assumption that $\bar{\mathbf{r}} \in \mathrm{int}\mathbb{D}_2^n$ throughout the rest of the paper.

For any polygon with edges (e_1, \dots, e_n) of the prescribed side lengths $\mathbf{r} = (r_1, \dots, r_n)$ such that $\bar{\mathbf{r}} \in \mathrm{int}\mathbb{D}_2^n$, set

$$u_i = \frac{e_i}{r_i} \text{ for } 1 \leq i \leq n.$$

Then

$$(u_1, \dots, u_n) \in (S^2)^n \cong (\mathbb{P}^1)^n$$

and satisfies

$$r_1 u_i + \dots + r_n u_n = e_1 + \dots + e_n = 0.$$

That is, $(u_1, \dots, u_n) \in \Phi_{\mathbf{r}}^{-1}(0)$. Conversely, making the above arguments backward, any point in $\Phi_{\mathbf{r}}^{-1}(0)$ determines a polygon with the prescribed side lengths $\mathbf{r} = (r_1, \dots, r_n)$. Thus we obtain the natural identification

$$M_{\mathbf{r}} \cong \Phi_{\mathbf{r}}^{-1}(0)/\mathrm{SO}(3)$$

which, via the identification

$$\Phi_{\mathbf{r}}^{-1}(0)/\mathrm{SO}(3) \cong (\mathbb{P}^1)_{ss}^n(\mathbf{r})//\mathrm{PGL}(2, \mathbb{C}),$$

provides $M_{\mathbf{r}}$ a holomorphic structure and a Kähler symplectic form $\omega_{\mathbf{r}}$ away from singularities.

2.3. Alternatively, the moduli space $M_{\mathbf{r}}$ can also be constructed as a symplectic quotient of the Grassmannian $G(2, \mathbb{C}^n)$ by the maximal torus $(\mathbb{C}^*)^n$. A simple way to see this is to look at the natural action of the group $(\mathbb{C}^*)^n \times GL(2, \mathbb{C})$ on the space of full-rank matrices of size $2 \times n$ and then take quotients by stage. Using this approach to $M_{\mathbf{r}}$, the results of [3] and [8] (among others) can be applied directly.

2.4. The hypersimplex is divided into chambers (maximal or otherwise, making a polytopal chamber complex) by the walls defined as follows

$$W_J = \{(x_1, \dots, x_n) \in \mathbb{D}_2^n : \sum_{i \in J} x_i = 1\}$$

where J runs over all the proper subsets of $[n] = \{1, 2, \dots, n\}$. J and its complement J^c define the same wall $W_J = W_{J^c}$. When $|J| = 1$ or $n - 1$, W_J is a facet of \mathbb{D}_2^n which is a *simplex*. Other facets of \mathbb{D}_2^n are of the form

$$F_i = \{(x_1, \dots, x_n) \in \mathbb{D}_2^n : x_i = 0\}.$$

When $n > 4$, F_i are again hypersimplices. A proper subset J defines an interior wall if and only if $2 \leq |J| \leq n - 2$ (i.e., $|J| > 1$ and $|J^c| > 1$). Each interior wall W_J divides \mathbb{D}_2^n into two parts

$$\{(x_1, \dots, x_n) \in \mathbb{D}_2^n : \sum_J x_j \leq 1\}$$

and

$$\{(x_1, \dots, x_n) \in \mathbb{D}_2^n : \sum_{J^c} x_j \leq 1\}.$$

Two points x and y are in the same chamber (maximal or otherwise) if the following holds. For all proper subset J ,

$$\sum_J x_j \leq 1 \iff \sum_J y_j \leq 1.$$

When only strict inequalities occur, we get characterizations of maximal chambers. Points in the same chamber define homeomorphic (actually biholomorphic) moduli spaces of polygons. But their naturally equipped structures of symplectic Kähler space are different, in general.

2.5. We should also consider the positive cone over \mathbb{D}_2^n to take into account n -gons with all possible side lengths

$$C(\mathbb{D}_2^n) := \{(x_1, \dots, x_n) \in \mathbb{R}_+^n : \sum_i x_i \neq 0, \frac{2(x_1, \dots, x_n)}{\sum_i x_i} \in \mathbb{D}_2^n\}$$

where \mathbb{R}_+ is the set of all nonnegative real numbers. Equivalently,

$$C(\mathbb{D}_2^n) \cup \{0\} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq \sum_{j \neq i} x_j, i = 1, \dots, n\}.$$

For any n -gon $P \in M_{\mathbf{r}}$, the length vector \mathbf{r} belongs to $C(\mathbb{D}_2^n)$, that is, $\bar{\mathbf{r}} \in \mathbb{D}_2^n$. Walls and chambers in \mathbb{D}_2^n induce obvious walls and chambers in $C(\mathbb{D}_2^n)$. Cone walls and cone chambers will generally be denoted by $C(W_J)$ and $C(\triangle)$ (etc.) if W_J is a wall of \mathbb{D}_2^n and \triangle is chamber of \mathbb{D}_2^n . All the above discussions extend to the cone $C(\mathbb{D}_2^n)$ directly. The dependence of the moduli spaces of polygons on the cone chambers is the obvious one.

2.6. Let \mathbf{r} and \mathbf{r}' be in the same chamber C (maximal or otherwise). C is rational and thus contains a rational point \mathbf{r}_C . Let M_C denote the GIT quotient $(\mathbb{P}^1)_{ss}^n(\mathbf{r}_C) // \mathrm{PGL}(2)$. This is a projective variety and depends only on the chamber C . Then we have the following isomorphisms

$$M_{\mathbf{r}} \longrightarrow M_C \longleftarrow M_{\mathbf{r}'}.$$

The two isomorphisms are *canonical* because they are naturally induced from the inclusions $\Phi_{\mathbf{r}}^{-1}(0) \subset (\mathbb{P}^1)_{ss}^n(\mathbf{r}_C) \supset \Phi_{\mathbf{r}'}^{-1}(0)$, which, of course, depend on no auxiliary choices. Hence, we obtain an induced *canonical* isomorphism

$$f_{\mathbf{r}\mathbf{r}'} : M_{\mathbf{r}} \rightarrow M_{\mathbf{r}'}.$$

Definition 2.7. A polygon is called a line gon if it lies entirely on a straight line.

A line gon occurs if and only if there is a proper subset J with

$$\sum_J r_i = \sum_{J^c} r_i.$$

When $n \geq 5$, they are singular points of the moduli space and are isolated.

2.8. Let C be a chamber that lies on the boundary of another chamber C' . By [11] (also [2] among others), there exists a canonical projective morphism

$$\beta_{C'C} : M_{C'} \rightarrow M_C$$

which, by 2.6, induces a canonical complex analytic map

$$\beta_{r'r} : M_{r'} \rightarrow M_r$$

where $r' \in C'$ and $r \in C$. When C' is a maximal chamber, the above maps are resolutions of singularities if $n > 4$.

3. FAVORABLE CHAMBERS

3.1. For chambers adjacent to the boundary of the hypersimplex, the corresponding moduli spaces take simple forms.

Let Δ_i be the (unique) maximal chamber in \mathbb{D}_2^n that contains the simplex facet $W_{\{i\}}$ ($1 \leq i \leq n$). Numerically, this chamber can be characterized as follows. $x \in \Delta_i$ if and only if

$$\sum_J x_j < 1$$

for all proper J with $i \notin J$ and $|J| < n - 1$. Or equivalently (taking the complement J^c),

$$\sum_J x_j + x_i > 1$$

for all proper J with $i \notin J$ if and only if

$$x_j + x_i > 1$$

for all $j \neq i$.

Definition 3.2. These chambers Δ_i ($i = 1, \dots, n$) and their corresponding cones $C(\Delta_i)$ will play special rôles in this paper. We will refer them as *favorable chambers*.

Proposition 3.3. (Theorem 2.1, [11] and Theorem 6.14, [12]) *Let \mathbf{r} be an element in the interior of the favorable chamber $C(\Delta_i)$. Then $M_{\mathbf{r}}$ is isomorphic to \mathbb{P}^{n-3} .*

Proposition 3.4. *Assume that $n \geq 3$. Let $\mathbf{r} \in C(\Delta_i)$. Then the i -th edge e_i is the longest edge, that is, $r_i > r_j$ for all $j \neq i$.*

Proof. Dividing by $\frac{2}{L(\mathbf{r})}$, we may assume that $\mathbf{r} \in \Delta_i$. Assume otherwise that $r_j \geq r_i$ for some j . Then

$$1 > \sum_{h \neq i} r_h = \sum_{h \neq i, j} r_h + r_j \geq \sum_{h \neq i, j} r_h + r_i > 1,$$

a contradiction. □

Recall from the introduction that for simplicity, edges are said to be parallel if they point the same direction. Edges pointing in opposite directions are considered to be anti-parallel, not parallel.

Definition 3.5. A polygon is said to degenerate at a set of edges $e_I := \{e_i\}_{i \in I}$ with $|I| > 1$ if $\{e_i\}_{i \in I}$ are all parallel and no other edges are parallel to them. An edge is said to be degenerate if it belongs to a set of degenerate edges, e_I with $|I| > 1$.

Proposition 3.6. *Let \mathbf{r} be an element in the interior of the favorable chamber $C(\triangle_i)$ and $P \in M_{\mathbf{r}}$. Then the i -th edge of P never degenerates along with others.*

Proof. Dividing by $\frac{2}{L(\mathbf{r})}$, we may assume that $\mathbf{r} \in \triangle_i$. Assume otherwise. Then

$$r_i + r_j \leq \sum_{k \neq i, j} r_k$$

for some $j \neq i$ which implies that $r_i + r_j \leq 1$, a contradiction. \square

Remark 3.7. Likewise, there is also a maximal chamber ∇_i in \mathbb{D}_2^n containing a (relatively) maximal chamber in the hypersimplex facet $F_{\{i\}}$ ($1 \leq i \leq n$) such that for any \mathbf{r} in the interior of the chamber ∇_i , $M_{\mathbf{r}}$ is isomorphic to $(\mathbb{P}^1)^{n-3}$. More precisely, a point x of \mathbb{D}_2^n belongs to the chamber ∇_i if and only if

$$\sum_J x_j > 1$$

for all proper J with $i \notin J$ if and only if

$$x_j + x_k > 1$$

for all $j, k \neq i$. Or equivalently (taking the complement J^c),

$$\sum_J x_j + x_i < 1$$

for all proper J with $i \notin J$. For more information about ∇_i , consult Theorem 2.1 of [11] and Theorem 6.10 of [12]. We point out that these particular chambers will not play any special rôles in this paper (although they may do if a different approach than the one in this paper is taken).

4. STABLE DEGENERATION OF POLYGONS

To clarify the principal subject of our study, we shall give in this section a detailed set-theoretic description of stable polygons, leaving out their finer structures to be treated in the sequel.

Definition 4.1. An n -gon is called generic if it does not have parallel edges. Given a side length vector \mathbf{r} , we use $M_{\mathbf{r}}^0$ to denote the subspace of generic polygons.

Lemma 4.2. *The moduli space $M_{\mathbf{r}}^0$ of all generic n -gons with the fixed side lengths \mathbf{r} can be identified with the moduli space of n distinct ordered points on the projective line.*

Proof. This follows from the identification $M_{\mathbf{r}} \cong (\mathbb{P}^1)_{ss}^n(\mathbf{r}) // \mathrm{PGL}(2)$. \square

This is an open holomorphic space (in fact, quasi-projective) and we are looking for nice and geometrically meaningful compactifications of it by adding *limiting polygonal objects* such that the added objects fit together to form a divisor with normal crossings. The limiting polygonal objects that we choose to add will be called stable polygons which we will describe now.

4.3. The idea is the usual one as in Deligne-Mumford's construction of stable pointed curves. We follow the spirit of Fulton-MacPherson [4]. The principal guiding principle is that whenever a set of edges become parallel, we will resolve it by introducing a "bubble" polygon in a coherent way. Here goes some detailed descriptions of stable n -gons for $n \leq 5$ to just gain some concrete feelings about general stable polygons.

All triangles are generic and thus stable.

A generic quadrangle is stable. Given a degenerate quadrangle P_0 with two parallel edges e_i and e_j , to remedy the "coincidence" problem, we introduce an independent arbitrary (stable) triangle P_1 with side lengths

$$(r_i, r_j, r_i + r_j - \epsilon_{i,j})$$

where $\epsilon_{i,j}$ is a fixed suitably³ small positive number. The collection (P_0, P_1) of these two labeled (but not ordered) polygons is a (reducible) stable quadrangle.

For pentagons, one degeneration case is just like quadrangles: two edges e_i and e_j are parallel. In this case, we introduce an independent

³This will be made precise when stable polygons are formally defined.

arbitrary (stable) triangle in exactly the same way as we did in the quadrangle cases.

When three edges, e_i , e_j , e_k , are parallel, to get a moduli-stable pentagon, we add an independent arbitrary moduli-stable quadrangle with prescribed side lengths

$$(r_i, r_j, r_k, r_i + r_j + r_k - \epsilon_{i,j,k})$$

where $\epsilon_{i,j,k}$ is a suitably small positive number. The collection of these labeled (but not ordered) polygons is a (reducible) stable pentagon.

4.4. In general, given any polygon, if k edges are parallel, we add a generic $(k+1)$ -gon whose first k sides inherit the lengths of the original edges but whose last side has a new length $r_{i_1} + \cdots + r_{i_k} - \epsilon_{i_1, \dots, i_k}$ with a choice of a small positive number $\epsilon_{i_1, \dots, i_k}$. In any of the polygons obtained, we allow more degeneration, and whenever edges become parallel, we introduce (bubble) polygons as above. It is important to point out that in the further degeneration, the edges with the new lengths⁴ will not degenerate (point the direction of any other edge). This will be automatic after choosing $\epsilon_{i_1, \dots, i_k}$ carefully (see Corollary 4.7 below).

To formally define an arbitrary stable polygon, some preparations are in order.

Lemma 4.5. *Let \mathbf{r} be any point in the interior of $C(\mathbb{D}_2^n)$. Then*

$$(\mathbf{r}, \sum r_i - \epsilon)$$

is in the chamber of $C(\Delta_{n+1})$ of $C(\mathbb{D}_2^{n+1})$ if and only if

$$\epsilon < 2 \min\{r_1, \dots, r_n\}.$$

Proof. By 3.1, we need

$$\frac{2}{2 \sum r_i - \epsilon} \sum_J r_j < 1$$

for all proper subsets J of $\{1, \dots, n\}$, which is equivalent to

$$\epsilon < 2 \sum_{J^c} r_i$$

for all proper subsets J of $\{1, \dots, n\}$, which is, in turn, equivalent to

$$\epsilon < 2 \min\{r_1, \dots, r_n\}.$$

□

⁴Such edges correspond to double points of Deligne-Mumford stable curves, as we shall see later.

4.6. Fix a vector \mathbf{r} in the interior of the cone $C(\mathbb{D}_2^n)$ over the hypersimplex \mathbb{D}_2^n . Since \mathbf{r} lies in the *interior* of the cone $C(\mathbb{D}_2^n)$, it is impossible to have $(n-1)$ many edges to point to the same direction. It does not make sense to treat the case that “there is one edge pointing to the same direction”. Less trivial and a bit tricky is the case when exactly two edges point the same direction. According to the rule, we will have to, in this case, introduce a triangle to make the polygon stable. However, because a triangle is rigid, it affects neither the moduli space nor the symplectic structure (§§5, 6 and 7). This last point will be important when we consider the Kähler cone of $\overline{\mathcal{M}}_{0,n}$.

So for any proper subset $J \subset \{1, \dots, n\}$ with cardinality $1 < |J| < n-1$ we fix a (suitably small) positive number

$$0 < \epsilon_J < 2 \min_J \{r_j\}.$$

Set

$$\mathbf{r}_{J, \epsilon_J} = (r_J, \sum_J r_j - \epsilon_J).$$

We frequently write $\mathbf{r}_{J, \epsilon_J}$ as \mathbf{r}_J when no confusion may occur.

Corollary 4.7. *The vector $\mathbf{r}_{J, \epsilon_J}$ lies in a favorable chamber. Consequently, $M_{\mathbf{r}_J}$ is isomorphic to some projective space and hence is independent of the choice of ϵ_J . Furthermore, no polygon in $M_{\mathbf{r}_J}$ ever degenerates at the last edge (i.e., the longest edge).*

Remark 4.8. Note that for any $0 < \epsilon < 2 \min\{r_1, \dots, r_n\}$, we can set $\epsilon_J = \epsilon$ for all proper subsets $J \subset [n]$. This will satisfy part of our purposes (cf. Corollary 4.7) and may save some notational mess. In particular, there exists a *canonical* choice

$$\epsilon_J = \min\{r_1, \dots, r_n\}$$

for all proper subsets $J \subset [n]$. To keep generality, however, we will work with an arbitrary choice of ϵ_J in the legal range.

Remark 4.9. We will use C_J to denote the favorable chamber that \mathbf{r}_J belongs to. In particular, M_{C_J} stands for the common projective model for all $M_{\mathbf{x}}$ with $\mathbf{x} \in C_J$. Note that this chamber does not depend on the choice of ϵ_J .

Definition 4.10. Let \mathbf{r} be a point in the interior of $C(\mathbb{D}_2^n)$. Given any proper subset J of $\{1, \dots, n\}$ with $|J| \geq 2$, a pair

$$(P, P') \in M_{\mathbf{r}} \times M_{\mathbf{r}_J}$$

is said to be a bubble pair if P degenerates at the edges e_J (i.e., the edges $\{e_j\}_J$ of P point to the same direction and no other edges point to the direction of these edges). In this case, P' is called a bubble of P .

Figure 4

By Corollary 4.7, P' never degenerates at the last edge (i.e., the longest edge whose length is $\sum_J r_j - \epsilon_J$). We point out that $(P, P') \in M_{\mathbf{r}} \times M_{\mathbf{r}_J}$ being a bubble pair implies that

$$\sum_J r_j \leq \sum_{J^c} r_j.$$

Definition 4.11. For this reason, we shall call such a proper subset *relevant*. Thus only *relevant* proper subset J will occur as the index sets for bubble pairs.

Thus the relevancy is a relative concept. This point will be useful in the definition of stable polygons. For simplicity, the set of all relevant subsets $J \subset \{1, \dots, n\}$ with respect to \mathbf{r} is denoted by $\mathcal{R}(\mathbf{r})$.

4.12. Now let \mathbf{r} be any point in the interior of $C(\mathbb{D}_2^n)$ and

$$\varepsilon = \{\epsilon_J | J \in \mathcal{R}(\mathbf{r})\}$$

with ϵ_J chosen as in 4.6. Fix them once and for all. (We point out again that by Remark 4.8, we may choose all ϵ_J equal to a fixed number ϵ . And there is even an canonical choice of such ϵ , namely, $\min\{r_1, \dots, r_n\}$.)

For a notational clarification, we make a remark that in this paper, a plain Greek letter ϵ is always to mean a suitably small positive number, while ε is to stand for a collection of such ϵ .

Definition 4.13. A stable n -gon with respect to the side length vector \mathbf{r} and $\varepsilon = \{\epsilon_J | J \in \mathcal{R}(\mathbf{r})\}$ is a collection of labeled (but *not ordered*) polygons

$$\mathbf{P} := (P_0, P_1, \dots, P_m) \in M_{\mathbf{r}} \times M_{\mathbf{r}_{J_1}} \times \dots \times M_{\mathbf{r}_{J_m}}$$

satisfying the following properties:

1. whenever $J_t \subset J_s$, then P_t is a bubble of P_s .
2. if P_h does not have a bubble, then it is generic (i.e., $P_h \in M_{\mathbf{r}_{J_h}}^0$).

In particular, in Definition 4.13 (1), we must have that $J_t \subset J_s$ is relevant with respect to \mathbf{r}_{J_s} .

One thing worths a word now. When $J \in \mathcal{R}(\mathbf{r})$ and $|J| = 2$, M_{r_J} consists of a single triangle. Although it is convenient to include it to define a stable polygon, in effect, it will not do anything to the moduli space. In particular, when we later formulate the Kähler cone of $\overline{\mathcal{M}}_{0,n}$, these "small" J 's have to be dropped out from computations. For later use, we set $\mathcal{R}_{>2}(\mathbf{r}) = \{J \in \mathcal{R}(\mathbf{r}) | |J| > 2\}$.

Remark 4.14. *Using a permutation among edges, we can always arrange the edges pointing to the same direction to be adjacent. The inverse of the permutation will allow us to get back what we start with. This observation will simplify our exposition at points.*

Figure 5 illustrates two polygons that are related by the permutation

$$(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12) \rightarrow (1\ 7\ 2\ 10\ 3\ 11\ 4\ 12\ 5\ 8\ 6\ 9).$$

The edges of the first polygon are numbered starting from the left-most edge with the direction \rightarrow . The edges of the second polygon are numbered in the same way.

Figure 5

Definition 4.15. We use $\mathfrak{M}_{\mathbf{r},\varepsilon}$ to denote the set of all stable polygons. For simplicity, one may abbreviate $\mathfrak{M}_{\mathbf{r},\varepsilon}$ as $\mathfrak{M}_{\mathbf{r}}$.

It is rather easy to see that the set $\mathfrak{M}_{\mathbf{r},\varepsilon}$ of all stable polygons with the prescribed side vector \mathbf{r} and a choice of ε carries a natural compact Hausdorff topology. This topology will be the underlying topology of the complex structure on $\mathfrak{M}_{\mathbf{r},\varepsilon}$ which we will construct in the next section.

5. MODULI SPACES $\mathfrak{M}_{\mathbf{r},\varepsilon}$ OF STABLE POLYGONS

Throughout, unless stated otherwise, \mathbf{r} denotes a fixed length vector away from the boundary of $C(\mathbb{D}_2^n)$ and C denotes the chamber (not necessarily maximal) that contains \mathbf{r} .

5.1. Consider the space of n -polygons with one free side:

$$Z_{[n]} := \left\{ \begin{array}{l} n\text{-gons with the non-zero fixed side length } r_1, \dots, r_{n-1} \\ \text{but the last side is free} \end{array} \right\}.$$

For any polygon P in $Z_{[n]}$, define $l(P)$ to be the length of the free side (i.e., the last side). Then this length function

$$l : Z_{[n]} \rightarrow \mathbb{R}$$

assumes the maximal value $r_1 + \dots + r_{n-1}$. For convenience, we may allow the length of the free side to be zero. Then l also assumes the minimal value 0.

Every level set of l provides a moduli space of polygons with the obvious prescribed side lengths. At the extremal, $f^{-1}(0)$ is the moduli space of $(n-1)$ -gons with the prescribed side lengths (r_1, \dots, r_{n-1}) ; while $f^{-1}(r_1 + \dots + r_{n-1})$ consists of a single line polygon.

We are interested in $M_{\mathbf{r}} = l^{-1}(r_n)$ where the length of the last side r_n is close to the maximum $r_1 + \dots + r_{n-1}$.

By Lemma 4.5, if

$$r_1 + \dots + r_{n-1} - 2 \min\{r_1, \dots, r_{n-1}\} < a < b < r_1 + \dots + r_{n-1},$$

then $l^{-1}(a)$ and $l^{-1}(b)$ are both isomorphic to the same projective space. We may use f_{ab} to denote the *canonical* isomorphism from $l^{-1}(a)$ to $l^{-1}(b)$:

$$f_{ab} : l^{-1}(a) \rightarrow l^{-1}(b).$$

When $b = r_1 + \dots + r_{n-1}$, $l^{-1}(b)$ is a single point. In this case, we understand f_{ab} as the total collapsing map $l^{-1}(a) \rightarrow l^{-1}(b)$.

5.2. Let $P_0 \in M_{\mathbf{r}}$ be an n -gon and $J = \{j_1 < \dots < j_s\}$ a subset of $\{1, \dots, n\}$ such that P_0 degenerates at e_J (see Definition 3.5). For any $P \in M_{\mathbf{r}}$, *via a permutation if necessary, we can assume that the edges e_{j_1}, \dots, e_{j_s} are adjacent in that order.* Now let

$$d_J = -(e_{j_1} + \dots + e_{j_s})$$

be the vector starting from the end of e_{j_s} and ending at the initial of e_{j_1} . d_J will be referred as the J -diagonal and can be zero in general. Then $e_j (j \in J)$ and d_J form a polygon Q_J , while via a similar arrangement, $-d_J$ and $e_j (j \in J^c)$ form a polygon Q_J^c . Informally, we may say that the J -diagonal d_J divides the (permuted) n -gon P into the union of two sub-polygons Q_J and Q_J^c : $P := Q_J \cup Q_J^c$. For our purpose, we assume that $l(Q_J)$ (=the length $|d_J|$ of the J -diagonal = the length of the last side of Q_J) is close to the maximal value of l in the sense that

$$r_{j_1} + \dots + r_{j_s} - 2 \min\{r_{j_1}, \dots, r_{j_s}\} < l(Q_J) \leq r_{j_1} + \dots + r_{j_s}.$$

Then $(r_{j_1}, \dots, r_{j_s}, l(Q_J))$ is in the favorable chamber C_J unless $l(Q_J) = r_{j_1} + \dots + r_{j_s}$ in which case the normalization of $(r_{j_1}, \dots, r_{j_s}, l(Q_J))$ belongs to a boundary simplex C'_J in \mathbb{D}_2^n .

Remark 5.3. Applying the inverse of the permutation (if necessary), we can remove the assumption that edges e_J are adjacent. For simplicity, rather than declaring this at every place where we use it, we will simply say that “the permutation scheme is applied” or “apply the permutation scheme”.

Definition 5.4. Let $P_0 \in M_{\mathbf{r}}$ be an n -gon and $J = \{j_1 < \dots < j_s\}$ a subset of $\{1, \dots, n\}$ such that P_0 degenerates at e_J . Let $P \in M_{\mathbf{r}}$ and $Q \in M_{\mathbf{r}, J}$. Let also Q_J and Q_J^c be as in 5.2. Here the permutation scheme is applied when necessary. We say that P and Q are incident, denoted by

$$\iota(P, Q),$$

if

$$r_{j_1} + \dots + r_{j_s} - 2 \min\{r_{j_1}, \dots, r_{j_s}\} < l(Q_J) = |d_J| \leq r_{j_1} + \dots + r_{j_s}$$

and Q is mapped to Q_J by the morphism $f_{l(Q)l(Q_J)}$. The map $f_{l(Q)l(Q_J)}$ is an isomorphism unless $l(Q_J)$ is maximal (i.e., Q_J is a line gon) in which case $f_{l(Q)l(Q_J)}$ is a total collapsing map to a point variety.

Figure 6 depicts an example where Q and Q_J are assumed to be isomorphic by the morphism $f_{l(Q)l(Q_J)}$.

Figure 6

Note that we automatically have for any $Q \in M_{\mathbf{r},J}$ that

$$r_{j_1} + \dots + r_{j_s} - 2 \min\{r_{j_1}, \dots, r_{j_s}\} < l(Q) < r_{j_1} + \dots + r_{j_s}$$

because $l(Q) = \sum_J r_j - \epsilon_J$, by definition. Thus, the above definition makes sense because of the discussion in 5.2.

Lemma 5.5. *The incidence relation is complex analytic. That is, the subset of $M_{\mathbf{r}} \times M_{\mathbf{r},J}$ defined by incidence relation is complex analytic.*

Proof. It basically follows from the fact that all maps $f_{l(Q')l(Q)}$ in the definition of incidence relation are complex analytic. \square

5.6. Let $P_0 \in M_{\mathbf{r}}$ be a polygon that degenerates at the edges e_J . Here, if necessary, we may apply the permutation scheme. Set $U_{\mathbf{r}}(P_0)_J \subset M_{\mathbf{r}}$ to be the subset of $M_{\mathbf{r}}$ consisting of polygons P such that

$$r_{j_1} + \dots + r_{j_s} - 2 \min\{r_{j_1}, \dots, r_{j_s}\} < |d_J| \leq r_{j_1} + \dots + r_{j_s}.$$

This is an open neighborhood of P_0 in $M_{\mathbf{r}}$.

Lemma 5.7. *We have*

1. *If P_0 is not a line gon, $U_{\mathbf{r}}(P_0)_J$ does not contain a line gon. In particular, $U_{\mathbf{r}}(P_0)_J$ is smooth.*
2. *If P_0 is a line gon, P_0 is the only line gon in $U_{\mathbf{r}}(P_0)_J$. In particular, $U_{\mathbf{r}}(P_0)_J$ is smooth away from P_0 .*

Proof. We apply the permutation scheme and assume that the edges e_J are adjacent. Let P'_0 be a line gon in $U_{\mathbf{r}}(P_0)_J$ that degenerates at edges e_I for some $I \neq J$. (Here, whether the edges e_I are adjacent is irrelevant to our proof.) In particular, we have $\sum_I r_i = \sum_{I^c} r_i$. Let d_J be the J -diagonal for P'_0 . Without loss of generality, assume that $\sum_{I \cap J} r_j \geq \sum_{I^c \cap J} r_j$. Then

$$|d_J| = \left| \sum_{I \cap J} r_j - \sum_{I^c \cap J} r_j \right| = \sum_J r_j - 2 \sum_{I^c \cap J} r_j \leq \sum_J r_j - 2 \min_J \{r_j\}.$$

This contradicts that $P'_0 \in U_{\mathbf{r}}(P_0)_J$. \square

We will need a general lemma

Lemma 5.8. (Removable Singularity Theorem). *Let $f : X \rightarrow Y$ be a continuous map between two holomorphic complex varieties. Assume that the restriction f_0 of f to a dense open subset of X is holomorphic. Then f is itself holomorphic.*

Proof. It follows from the consideration of the following diagram

$$f : X \rightarrow \text{Graph}(f) = \overline{\text{Graph}(f_0)} \rightarrow X \times Y \rightarrow Y.$$

□

When P_0 is a line gon that degenerate at e_J and e_{J^c} , then e_J and e_{J^c} are the only sets of degenerating edges for P_0 . Set $U_{\mathbf{r}}^0(P_0) = U_{\mathbf{r}}(P_0)_J \cap U_{\mathbf{r}}(P_0)_{J^c}$. This is again an open (singular) neighborhood of P_0 in $M_{\mathbf{r}}$. Define a correspondence set

$$\widehat{U}_{\mathbf{r}}^1(P_0) \subset U_{\mathbf{r}}^0(P_0) \times M_{\mathbf{r}_J} \times M_{\mathbf{r}_{J^c}}$$

as follows

$$\widehat{U}_{\mathbf{r}}^1(P_0) = \{(P, P_1, P_2) \in M_{\mathbf{r}} \times M_{\mathbf{r}_J} \times M_{\mathbf{r}_{J^c}} : \iota(P, P_1), \iota(P, P_2)\}.$$

$\widehat{U}_{\mathbf{r}}^1(P_0)$ projects onto $U_{\mathbf{r}}^0(P_0)$ by forgetting the last two factors.

Lemma 5.9. *Let P_0 be a line gon. Then the projection*

$$\widehat{U}_{\mathbf{r}}^1(P_0) \rightarrow U_{\mathbf{r}}^0(P_0)$$

is a (canonical) resolution of singularities⁵ of $U_{\mathbf{r}}^0(P_0)$.

Proof. We will apply the permutation scheme in this proof. $\widehat{U}_{\mathbf{r}}^1(P_0)$ is the subset of $U_{\mathbf{r}}^0(P_0) \times M_{\mathbf{r}_J} \times M_{\mathbf{r}_{J^c}}$ defined by the incidence relation. It projects surjectively onto the subset $U_{\mathbf{r}}(J^c)$ of $U_{\mathbf{r}}^0(P_0) \times M_{\mathbf{r}_{J^c}}$ by forgetting the middle factor. $U_{\mathbf{r}}(J^c)$ is nothing but

$$U_{\mathbf{r}}(J^c) = \{(P, P_1) \in U_{\mathbf{r}}^0(P_0) \times M_{\mathbf{r}_{J^c}} : \iota(P, P_1)\}.$$

We will first argue that the subset $U_{\mathbf{r}}(J^c)$ is smooth by identifying it with a smooth open subset of $M_{\mathbf{r}'}$ where \mathbf{r}' is a perturbation of \mathbf{r} . For this purpose, assume that

$$\mathbf{r}_J = (r_{j_1}, \dots, r_{j_s}, \sum_J r_j - \epsilon_J), \text{ and } \mathbf{r}_{J^c} = (r_{J^c}, \sum_{J^c} r_j - \epsilon_{J^c}).$$

Note that

$$\sum_J r_j = \sum_{J^c} r_j.$$

Let $\mathbf{r}'_J = (r'_{j_1}, \dots, r'_{j_s}, \sum_J r_j - \epsilon) \in C_J$ be a small generic perturbation of $\mathbf{r}_J = (r_{j_1}, \dots, r_{j_s}, \sum_J r_j - \epsilon_J)$ in C_J such that

$$r'_{j_1} + \dots + r'_{j_s} = \sum_{J^c} r_j - \epsilon_{J^c}.$$

⁵This lemma is a special case of a more general result from the Geometric Invariant Theory or the theory of symplectic reductions. Our proof is independent and uses only polygons.

Let \mathbf{r}' to be obtained from \mathbf{r} by replacing r_{j_1}, \dots, r_{j_s} by $r'_{j_1}, \dots, r'_{j_s}$ correspondingly. Since \mathbf{r}' is in more general position, and can be taken to be close to \mathbf{r} , we get a canonical surjection $\beta : M_{\mathbf{r}'} \rightarrow M_{\mathbf{r}}$ by 2.8.

Let $U_{\mathbf{r}'} = \beta^{-1}(U_{\mathbf{r}}^0(P_0))$. Then β restricts to an isomorphism between $U_{\mathbf{r}'} \setminus \beta^{-1}(P_0)$ and $U_{\mathbf{r}}^0(P_0) \setminus \{P_0\}$. Note that $\beta^{-1}(P_0)$ consists of the polygons of $U_{\mathbf{r}'}$ that degenerate at the edges e_J .

Define

$$g : U_{\mathbf{r}}(J^c) \rightarrow U_{\mathbf{r}'}$$

as

$$\begin{aligned} g : (P, P_1) &\rightarrow \beta^{-1}(P), \text{ if } P \neq P_0; \\ g : (P_0, P_1) &\rightarrow P', \text{ if } P = P_0 \end{aligned}$$

where P' is obtained from $P_1 \in M_{\mathbf{r}_{J^c}}$ by breaking the longest edge (whose length is $\sum_{J^c} r_j - \epsilon_{J^c}$) into the consecutive edges according to the lengths $r'_{j_1}, \dots, r'_{j_s}$ (which add up to $\sum_{J^c} r_j - \epsilon_{J^c}$, by assumption). The inverse of g can be checked to be

$$g^{-1} : P' \rightarrow (\beta(P'), P_1), \text{ if } \beta(P') \neq P_0$$

where $P_1 \in M_{\mathbf{r}_{J^c}}$ is uniquely determined by $\beta(P')$.

$$g^{-1} : P' \rightarrow (\beta(P'), P_1), \text{ if } \beta(P) = P_0$$

where P_1 is obtained from P' (which degenerates at e_J) by taking the degenerating edges as a single edge whose length is, by assumption, $\sum_J r'_j = \sum_{J^c} r_j - \epsilon_{J^c}$. The function g and g^{-1} are easily seen to be continuous and holomorphic on some dense open subsets. Thus by the Removable Singularity Theorem (Lemma 5.8), g and g^{-1} are both holomorphic.

By the same proof as in Lemma 5.7, $U_{\mathbf{r}'}$ contains no line gons (it also more or less obviously follows from the fact that $U_{\mathbf{r}'} = \beta^{-1}(U_{\mathbf{r}}^0(P_0))$ and Lemma 5.7). This shows that $U_{\mathbf{r}}(J^c) \cong U_{\mathbf{r}'}$ is smooth. Consider the projection

$$\widehat{U}_{\mathbf{r}}^1(P_0) \rightarrow U_{\mathbf{r}}(J^c).$$

It is easy to see that all fibers of the projection are projective spaces (mostly \mathbb{P}^0) and the projection is bimeromorphic. Thus the uniqueness of blowup implies that $\widehat{U}_{\mathbf{r}}^1(P_0) \rightarrow U_{\mathbf{r}}(J^c)$ is a blowup of $U_{\mathbf{r}}(J^c)$ along smooth centers. Thus $\widehat{U}_{\mathbf{r}}^1(P_0)$ is smooth, and this completes the proof. \square

Remark 5.10. Likewise, one obtains $U_{\mathbf{r}}(J)$ by forgetting the last factor of

$$\widehat{U}_{\mathbf{r}}^1(P_0) \subset U_{\mathbf{r}}^0(P_0) \times M_{\mathbf{r}_J} \times M_{\mathbf{r}_{J^c}}.$$

By the same arguments as in the above proof, $U_{\mathbf{r}}(J)$ can be identified with a smooth open subset of $M_{\mathbf{r}''}$ for some other \mathbf{r}'' . It follows then that $U_{\mathbf{r}}(J^c) \rightarrow U_{\mathbf{r}}^0(P_0)$ and $U_{\mathbf{r}}(J) \rightarrow U_{\mathbf{r}}^0(P_0)$ are also resolution of singularities, although neither is canonical due to a choice between J and J^c . As a consequence of the above, we have shown that $\widehat{U}_{\mathbf{r}}^1(P_0)$ is a common blowup of $U_{\mathbf{r}}(J^c)$ and $U_{\mathbf{r}}(J)$.

If we set Z_J (compare with $Z_{[n]}$) to be the moduli space of $(|J| + 1)$ -gons with first $|J|$ sides having fixed length r_{j_1}, \dots, r_{j_s} but the last side being free, then all the above discussions about $Z_{[n]}$ apply to Z_J in an obvious way.

5.11. This allows us to define the following correspondence (variety). For any stable polygon

$$\mathbf{P} = (P_0, P_1, \dots, P_m) \in M_{\mathbf{r}} \times M_{\mathbf{r}_{J_1}} \times \dots \times M_{\mathbf{r}_{J_m}},$$

define

$$U_{\mathbf{r}}(\mathbf{P}) = \left\{ \begin{array}{l} (P'_0, P'_1, \dots, P'_m) \in M_{\mathbf{r}} \times M_{\mathbf{r}_{J_1}} \times \dots \times M_{\mathbf{r}_{J_m}} : \\ \iota(P'_s, P'_t) \text{ whenever } J_t \subset J_s \end{array} \right\}.$$

If $\mathbf{P} = P_0$, we simply set $U_{\mathbf{r}}(\mathbf{P}) = M_{\mathbf{r}}^0$. Note that the components of a stable polygon are labeled but not ordered. Consequently, we need to point out that the incident correspondence $U_{\mathbf{r}}(\mathbf{P})$ is, up to isomorphisms (induced by permutations among the components), uniquely determined by the stable polygon \mathbf{P} .

Lemma 5.12. *Let $(P'_0, P'_1, \dots, P'_m) \in U_{\mathbf{r}}(\mathbf{P})$. For every $J_t \subset J_s$, P'_t is uniquely determined by P'_s unless P'_t is a bubble of P'_s .*

Proof. This follows from the definition of incidence relation. □

It then follows

Corollary 5.13. *There is a canonical injection*

$$\gamma_{\mathbf{r}}(\mathbf{P}) : U_{\mathbf{r}}(\mathbf{P}) \rightarrow \mathfrak{M}_{\mathbf{r}, \varepsilon}.$$

Lemma 5.14. *Given any stable polygon \mathbf{P} , $U_{\mathbf{r}}(\mathbf{P})$ is a smooth complex analytic subvariety of $M_{\mathbf{r}} \times M_{\mathbf{r}_{J_1}} \times \dots \times M_{\mathbf{r}_{J_m}}$.*

Proof. In this proof, the permutation scheme will be applied whenever the length of a diagonal of a polygon is used. Given any stable polygon

$$\mathbf{P} = (P_0, P_1, \dots, P_m) \in M_{\mathbf{r}} \times M_{\mathbf{r}_{J_1}} \times \dots \times M_{\mathbf{r}_{J_m}},$$

we shall describe $U_{\mathbf{r}}(\mathbf{P})$ inductively. If P_0 is generic, $U_{\mathbf{r}}(\mathbf{P}) = M_{\mathbf{r}}^0$ is obviously complex analytic and smooth. So we assume that $m \geq 1$.

If $J = J_k$ for some $1 \leq k \leq m$, we will call $P'_J := P'_k$ to be the J -component (or, k -component) of $\mathbf{P}' = (P'_0, P'_1, \dots, P'_m)$.

First, let $U_{\mathbf{r}}^0(\mathbf{P})$ be the subset of $M_{\mathbf{r}}$ consisting of points P'_0 with the following property: For any J such that P_0 degenerates at J , we require that the J -diagonal of P' satisfies

$$\sum_J r_j - 2 \min_J \{r_j\} < |d_J| \leq \sum_J r_j.$$

This is a complex analytic open neighborhood of P_0 in $M_{\mathbf{r}}$.

Define an incident correspondence $\widehat{U}_{\mathbf{r}}^1(\mathbf{P}) \subset U_{\mathbf{r}}^0(\mathbf{P}) \times \Pi_J M_{\mathbf{r}_J}$ as follows. Here, the product is over all J 's such that P_0 degenerates at the edges e_J . Such J 's will be referred as the subsets of the first kind. We then require that for any point $\mathbf{P}' \in \widehat{U}_{\mathbf{r}}^1(\mathbf{P})$ we have the incidence relation $i(P'_0, P'_J)$ where P'_0 is the (main) 0-component of \mathbf{P}' and P'_J is the J -component of \mathbf{P}' . Since the incidence relation is holomorphic, $\widehat{U}_{\mathbf{r}}^1(\mathbf{P})$ is holomorphic.

Then we take $U_{\mathbf{r}}^1(\mathbf{P})$ to be the subset of $\widehat{U}_{\mathbf{r}}^1(\mathbf{P})$ consisting of points with the following additional requirements. Given any point \mathbf{P}' in $\widehat{U}_{\mathbf{r}}^1(\mathbf{P})$, it belongs to $U_{\mathbf{r}}^1(\mathbf{P})$ if for any J -component P'_J of the point (where J is of the first kind) and any $I \subset J$ such that P'_J degenerates at the edges e_I , we demand that

$$\sum_I r_j - 2 \min_I \{r_i\} < l(d'_{J,I}) \leq \sum_{J \setminus I} r_j$$

where $d'_{J,I}$ is the I -diagonal of P'_J . Then $U_{\mathbf{r}}^1(\mathbf{P})$ is a holomorphic open subset of $\widehat{U}_{\mathbf{r}}^1(\mathbf{P})$. We may refer to the above I 's as the subsets of the second kind.

Likewise, one can define the incident correspondence variety $\widehat{U}_{\mathbf{r}}^2(\mathbf{P}) \subset U_{\mathbf{r}}^1(\mathbf{P}) \times \Pi_I M_{\mathbf{r}_I}$ where the product is over all subsets of the second kind and a complex analytic open subset $U_{\mathbf{r}}^2(\mathbf{P}) \subset \widehat{U}_{\mathbf{r}}^2(\mathbf{P})$, \dots , and keep going until all J_1, \dots, J_m are taken into account. We hence obtain inductively a sequence of projections

$$\begin{aligned} \widehat{U}_{\mathbf{r}}^1(\mathbf{P}) &\rightarrow U_{\mathbf{r}}^0(\mathbf{P}), \\ \widehat{U}_{\mathbf{r}}^2(\mathbf{P}) &\rightarrow U_{\mathbf{r}}^1(\mathbf{P}), \\ &\dots \\ \widehat{U}_{\mathbf{r}}^{h-1}(\mathbf{P}) &\rightarrow U_{\mathbf{r}}^{h-2}(\mathbf{P}), \\ \widehat{U}_{\mathbf{r}}^h(\mathbf{P}) &\rightarrow U_{\mathbf{r}}^{h-1}(\mathbf{P}). \end{aligned}$$

One checks directly that the so-inductively defined $\widehat{U}_{\mathbf{r}}^h(\mathbf{P})$ coincides with the incident correspondence variety $U_{\mathbf{r}}(\mathbf{P})$ as defined in 5.11.

To see that $U_{\mathbf{r}}(\mathbf{P})$ is smooth, we will analyze the above sequence. We will have to divide the proof into two cases.

Case 1. P_0 is not a line gon. In this case, P_0 is a smooth point of $M_{\mathbf{r}}$ and thus the open neighborhood $U_{\mathbf{r}}^0(\mathbf{P})$ is smooth by Lemma 5.7. Every fiber of the projection $\widehat{U}_{\mathbf{r}}^1(\mathbf{P}) \rightarrow U_{\mathbf{r}}^0(\mathbf{P})$ is easily seen to be projective spaces (mostly \mathbb{P}^0). In addition, the projection is bimeromorphic. Thus the uniqueness of blowup implies that the above projection is a blowup of $U_{\mathbf{r}}^0(\mathbf{P})$ along smooth centers. Indeed, the centers are the loci of $P'_0 \in U_{\mathbf{r}}^0(P_0)$ that are degenerate at some edges e_J . Each component of the center can be identified, by taking the degenerate edges e_J as a single edge, with a smooth open subset of $M_{\mathbf{x}}$ (which must be a projective space because such \mathbf{x} must be in a favorable chamber). Thus $\widehat{U}_{\mathbf{r}}^1(\mathbf{P})$ is smooth. Likewise, step by step, one can show that $\widehat{U}_{\mathbf{r}}^2(\mathbf{P})$, \dots , and $\widehat{U}_{\mathbf{r}}^h(\mathbf{P})$ are all smooth. In particular, $U_{\mathbf{r}}(\mathbf{P})$ is smooth.

Case 2. P_0 is a line gon. In this case, P_0 is a singular point of $M_{\mathbf{r}}$ and thus the open neighborhood $U_{\mathbf{r}}^0(\mathbf{P})$ is singular. By Lemma 5.9, noting that $\widehat{U}_{\mathbf{r}}^1(\mathbf{P}) = \widehat{U}_{\mathbf{r}}^1(P_0)$, the projection

$$\widehat{U}_{\mathbf{r}}^1(\mathbf{P}) \rightarrow U_{\mathbf{r}}^0(\mathbf{P})$$

is a (canonical) resolution of singularities of $U_{\mathbf{r}}^0(\mathbf{P})$.

Now, follow the proof in *Case 1*, build up from $\widehat{U}_{\mathbf{r}}^1(\mathbf{P})$, step by step, we conclude that $\widehat{U}_{\mathbf{r}}^h(\mathbf{P})$ is smooth as desired. \square

As an immediate consequence of Lemma 5.14, we have

Theorem 5.15. $\mathfrak{M}_{\mathbf{r},\varepsilon}$ carries a natural smooth, compact complex analytic structure induced by the injections $\gamma_{\mathbf{r}}(\mathbf{P}) : U_{\mathbf{r}}(\mathbf{P}) \rightarrow \mathfrak{M}_{\mathbf{r},\varepsilon}$ for all stable polygons \mathbf{P} .

Proof. The complex structures on $\gamma_{\mathbf{r}}(\mathbf{P})(U_{\mathbf{r}}(\mathbf{P}))$ (induced by those on $U_{\mathbf{r}}(\mathbf{P})$) for various stable polygons \mathbf{P} obviously agree with each other over the overlaps. The theorem then follows. \square

6. $\mathfrak{M}_{\mathbf{r},\varepsilon}$ AS ITERATED BLOWUPS OF $M_{\mathbf{r}}$

Globally, by forgetting all the bubbles of a stable polygon, or locally, by the projection from $U_{\mathbf{r}}(\mathbf{P})$ (see 5.11) to (the main factor) $M_{\mathbf{r}}$, we obtain

Corollary 6.1. *There is a canonical complex analytic map*

$$\pi_{\mathbf{r},\varepsilon} : \mathfrak{M}_{\mathbf{r},\varepsilon} \rightarrow M_{\mathbf{r}}$$

which restricts to the identity on $M_{\mathbf{r}}^0$.

Proof. The existence of the projection $\pi_{\mathbf{r}}$ as a set-theoretic map is obvious. That $\pi_{\mathbf{r}}$ is complex analytic follows from that locally $\pi_{\mathbf{r}}$ is equivalent to the projection $U_{\mathbf{r}}(\mathbf{P}) \rightarrow M_{\mathbf{r}}$. \square

6.2. When $M_{\mathbf{r}}$ is smooth, that is, when \mathbf{r} is away from walls, the map

$$\pi_{\mathbf{r},\varepsilon} : \mathfrak{M}_{\mathbf{r},\varepsilon} \rightarrow M_{\mathbf{r}}$$

ought to be an iterated blowup of $M_{\mathbf{r}}$ along some smooth subvarieties. The details go as follows.

6.3. Let $F[n]$ be the set of all partitions of $[n] = \{1, \dots, n\}$, partially ordered by reverse refinement. That is, an element of $F[n]$ is of the form $\alpha = I_1 \coprod \dots \coprod I_k = [n]$. Let $\beta = J_1 \coprod \dots \coprod J_m = [n]$ be another partition of $[n]$. We say $\alpha \leq \beta$ if for any $1 \leq s \leq m$, $J_s \subset I_t$ for some $1 \leq t \leq k$. The maximal element in $F[n]$ is $\{1\} \coprod \dots \coprod \{n\}$. The smallest element is $[n]$ itself (but we will never use this smallest element).

Given any α in $F[n]$, define Y_{α} to be the set of all polygons P in $M_{\mathbf{r}}$ such that the edges e_{I_s} ($1 \leq s \leq k$) are parallel. This is a closed subvariety of $M_{\mathbf{r}}$ which is isomorphic to $M_{\mathbf{r}_{\alpha}}$ by an obvious natural map⁶

$$f_{\alpha} : Y_{\alpha} \rightarrow M_{\mathbf{r}_{\alpha}},$$

where

$$\mathbf{r}_{\alpha} = \left(\sum_{s \in I_1} |e_s|, \dots, \sum_{s \in I_k} |e_s| \right).$$

Figure 7 illustrates the map f_{α} as the composition of a permutation followed by an identification.

Figure 7

⁶To see this, one may need to use permutations among edges. Consult Figure 7.

We must point out that the strata Y_α are empty for many $\alpha \in F[n]$ (e.g., when \mathbf{r}_α lies outside of the cone $C(\mathbb{D}_2^k)$).

Obviously, all Y_α are smooth if $M_{\mathbf{r}}$ is (having no line gons). In general, it always contains a smooth dense open subset

$$Y_\alpha^0 = f_\alpha^{-1}(M_{\mathbf{r}_\alpha}^0).$$

That is, Y_α^0 is the set of all polygons P in $M_{\mathbf{r}}$ such that the edges e_{I_s} ($1 \leq s \leq k$) are all parallel to each other but not to any other edges.

Then we obtain a canonical decomposition

$$M_{\mathbf{r}} = \bigcup_{\alpha \in F[n]} Y_\alpha^0.$$

This is a stratification of $M_{\mathbf{r}}$ by smooth strata.

One checks readily that

Proposition 6.4. *$Y_\alpha \subset Y_\beta$ if and only if $\alpha \leq \beta$.*

In fact, it can be shown that an intersection of the closed strata Y_α 's is again a closed stratum unless it is empty.

Now we come to our main theorems in this section.

Theorem 6.5. *Assume that $M_{\mathbf{r}}$ is smooth. Then*

$$\pi_{\mathbf{r},\varepsilon} : \mathfrak{M}_{\mathbf{r},\varepsilon} \rightarrow M_{\mathbf{r}}$$

is the iterated blowup of $M_{\mathbf{r}}$ along (the proper transforms of) all the smooth closed strata Y_α in the order dictated by the partial order ($Y_\alpha \subset Y_\beta$ if and only if $\alpha \leq \beta$), starting from the smallest ones.

Proof. This follows from the details of the proof presented for Lemma 5.14 and the fact that the blowup construction is local and unique. \square

Remark 6.6. For suitable ε , $\mathfrak{M}_{\mathbf{r},\varepsilon}$ comes equipped with a symplectic form $\Omega_{\mathbf{r},\varepsilon}$ whose cohomology class is uniquely defined so that

$$\pi_{\mathbf{r},\varepsilon} : (\mathfrak{M}_{\mathbf{r},\varepsilon}, \Omega_{\mathbf{r},\varepsilon}) \rightarrow (M_{\mathbf{r}}, \omega_{\mathbf{r}})$$

can be interpreted as a symplectic blowup (see Lemma 6.44 and pp 230 – 231 of [20]). The ambiguity of the so-called symplectic blowups provides another angle to see how natural it is the choices of ε in the definition of stable polygons.

The singular case is slightly complicated. Assume that $M_{\mathbf{r}}$ is singular. Then $M_{\mathbf{r}}$ has isolated singularities defined by line gons. Each of these line gons forms one of the smallest strata in $M_{\mathbf{r}} = \bigcup_{\alpha \in F[n]} Y_\alpha^0$. Recall by Lemma 5.9, these singularities admit canonical resolutions.

Theorem 6.7. *Assume that $M_{\mathbf{r}}$ is singular. Then*

$$\pi_{\mathbf{r},\varepsilon} : \mathfrak{M}_{\mathbf{r},\varepsilon} \rightarrow M_{\mathbf{r}}$$

is the composite of the canonical resolutions as described in Lemma 5.9 followed by the iterated blowups of the resulting resolution along (the proper transforms of) all other smooth closed strata Y_{α} in the order dictated by the partial order ($Y_{\alpha} \subset Y_{\beta}$ if and only if $\alpha \leq \beta$), starting from the smallest ones.

Proof. After applying Lemma 5.9, all other arguments remain the same as for the proof of Theorem 6.5 \square

Remark 6.8. When \mathbf{r} is on a wall, $M_{\mathbf{r}}$ carries a singular Kähler form. In this case, $\mathfrak{M}_{\mathbf{r},\varepsilon} \rightarrow M_{\mathbf{r}}$ ought to be interpreted as a Kähler morphism in a suitable sense.

6.9. Among all the iterated blowups

$$\pi_{\mathbf{r},\varepsilon} : (\mathfrak{M}_{\mathbf{r},\varepsilon}, \Omega_{\mathbf{r},\varepsilon}) \rightarrow (M_{\mathbf{r}}, \omega_{\mathbf{r}}),$$

two special cases worth mentioning. For one kind of special choices of \mathbf{r} , $M_{\mathbf{r}}$ is isomorphic to $(\mathbb{P}^1)^{n-3}$ (see 3.7). In this case, our presentation of the blowup $\mathfrak{M}_{\mathbf{r},\varepsilon} \rightarrow M_{\mathbf{r}}$, putting aside symplectic structures and after showing in the next section the isomorphism between $\mathfrak{M}_{\mathbf{r},\varepsilon}$ and $\overline{\mathcal{M}}_{0,n}$, specializes to the blowup representation of $\overline{\mathcal{M}}_{0,n}$, $\overline{\mathcal{M}}_{0,n} \rightarrow (\mathbb{P}^1)^{n-3}$, as utilized by Keel in his study on the Chow ring of $\overline{\mathcal{M}}_{0,n}$ ([15]). For some other special choices of \mathbf{r} , $M_{\mathbf{r}}$ is isomorphic to \mathbb{P}^{n-3} (see 3.3). In this case, our blowup $\mathfrak{M}_{\mathbf{r},\varepsilon} \rightarrow M_{\mathbf{r}}$, again forgetting symplectic structures, amounts to the blowup representation of $\overline{\mathcal{M}}_{0,n}$, $\overline{\mathcal{M}}_{0,n} \rightarrow \mathbb{P}^{n-3}$, as studied by Kapranov in [14]. We give below some details of the latter.

Example 6.10. Choose $\mathbf{r} \in \Delta_i$ for some fixed $1 \leq i \leq n$ (see §3 for the characterization of Δ_i). Note that a polygon $P \in M_{\mathbf{r}} \cong \mathbb{P}^{n-3}$ will never degenerate at the edge e_i . This fact together with the identification $f_{\alpha} : Y_{\alpha} \rightarrow M_{\mathbf{r}_{\alpha}}$ implies that every stratum Y_{α} is isomorphic to $\mathbb{P}^{\dim Y_{\alpha}}$. Point strata correspond to polygons P whose edges have exactly three different directions. There are $(n-1)$ such strata. They are points in \mathbb{P}^{n-3} in general position. Any other stratum, obviously containing a subset of these points, is the projective subspace spanned by the points in the subset. Applying Theorem 6.5 (together with the isomorphism $\mathfrak{M}_{\mathbf{r},\varepsilon} \cong \overline{\mathcal{M}}_{0,n}$ to be proved in §7), we recover Kapranov's blowup representation, $\overline{\mathcal{M}}_{0,n} \rightarrow \mathbb{P}^{n-3}$.

7. $\mathfrak{M}_{\mathbf{r},\varepsilon}$ AND $\overline{\mathcal{M}}_{0,n}$

7.1. Recall that a n -pointed connected complex algebraic curve of genus 0 is stable if

1. the n -marked points are smooth points;
2. every singular point is an ordinary double point;
3. for each irreducible connected component, the number of marked points plus the number of singular points on the component is at least 3.

The set $\overline{\mathcal{M}}_{0,n}$ of equivalence classes of all n -pointed stable algebraic curve of genus 0 carries a natural structure of a smooth projective variety. Let $\mathcal{M}_{0,n}$ be the moduli space of n -pointed smooth curves of genus zero. Then $\overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$ is a divisor with normal crossings. Given any n -pointed stable curve X , we can associate to it a graph $\text{graph}(X)$: the vertices of $\text{graph}(X)$ correspond to the irreducible components of X , and two vertices are joined by an edge if their corresponding components share a common singular point. Then that the curve X is of genus 0 is equivalent to the graph $\text{graph}(X)$ being a tree.

7.2. Likewise, we can also attach a graph $\text{graph}(\mathbf{P})$ to any given stable n -gon $\mathbf{P} = (P_0, P_1, \dots, P_m)$: the vertices of $\text{graph}(X)$ correspond to the polygons $\{P_0, P_1, \dots, P_m\}$, and two vertices are joined by an edge if their corresponding polygons satisfy the bubble relation (Definition 4.10). One checks easily that $\text{graph}(\mathbf{P})$ is a tree.

Theorem 7.3. *Let \mathbf{r} be a point in the interior of $C(\mathbb{D}_2^n)$ and ε be chosen as before. Then, $\mathfrak{M}_{\mathbf{r},\varepsilon}$ and $\overline{\mathcal{M}}_{0,n}$ are biholomorphic. Consequently, the complex structure on $\mathfrak{M}_{\mathbf{r},\varepsilon}$ is independent⁷ of \mathbf{r} and the choice of ε . Moreover, $\mathfrak{M}_{\mathbf{r},\varepsilon} \setminus M_{\mathbf{r}}^0$ is a divisor with normal crossings.*

Proof. $\mathfrak{M}_{\mathbf{r},\varepsilon}$ is easily seen to be bimeromorphic to $\overline{\mathcal{M}}_{0,n}$. That is, we have a biholomorphism

$$\gamma^0 : M_{\mathbf{r}}^0(\subset \mathfrak{M}_{\mathbf{r},\varepsilon}) \rightarrow \mathcal{M}_{0,n}(\subset \overline{\mathcal{M}}_{0,n}).$$

This map can be extended continuously as follows. For any stable polygon

$$\mathbf{P} = (P_0, P_1, \dots, P_m) \in M_{\mathbf{r}} \times M_{\mathbf{r}_{J_1}} \times \dots \times M_{\mathbf{r}_{J_m}},$$

we obtain a collection $X = (X_1, \dots, X_m)$ of pointed curves of genus zero via the identifications

$$M_{\mathbf{r}} \cong (\mathbb{P}^1)_{ss}^n(\mathbf{r}_C) // \text{PGL}(2).$$

⁷ The structure of Kähler space, however, depends on choices.

Now regarding the coinciding points in each X_i as a single point, and if (P_s, P_t) is a bubble pair, we joint X_s and X_t at the coinciding points (regarded as single point) of X_s (which corresponds to the same direction pointing edges of the polygon P_s) and the point of X_t that corresponds to the longest edge of P_t . This way, we obtain a reducible algebraic curve $X_1 \cup \dots \cup X_m$ which has n -labeled (smooth) points. It is of genus zero because its associated graph coincides with the graph of the stable polygon \mathbf{P} which is a tree by the construction. One checks that each component X_i has at least three distinguished (marked plus singular) points. Hence $X_1 \cup \dots \cup X_m$ is a stable n -pointed curve of genus zero. The so-induced map

$$\gamma : \mathfrak{M}_{\mathbf{r}, \varepsilon} \rightarrow \overline{\mathcal{M}}_{0, n}$$

$$(P_1, \dots, P_m) \rightarrow X_1 \cup \dots \cup X_m$$

is (more or less) obviously continuous and injective. By Lemma 5.8, γ is holomorphic. Since γ is bimeromorphic, it is also surjective. This implies that γ is biholomorphic. The rest follows from some well-known properties of $\overline{\mathcal{M}}_{0, n}$. \square

Remark 7.4. When \mathbf{r} is away from walls, being Kähler and Moishezon, $\mathfrak{M}_{\mathbf{r}, \varepsilon}$ is projective. Since $\mathfrak{M}_{\mathbf{r}, \varepsilon}$ and $\overline{\mathcal{M}}_{0, n}$ are biholomorphic, the GAGA theorems imply that $\mathfrak{M}_{\mathbf{r}, \varepsilon}$ and $\overline{\mathcal{M}}_{0, n}$ are isomorphic as projective varieties as well.

8. THE KÄHLER CONE OF $M_{\mathbf{r}}$

8.1. To pave a way to determining the Kähler cone of $\overline{\mathcal{M}}_{0, n}$, we first study the Kähler cone of $M_{\mathbf{r}}$. We shall mainly focus our attention on the most important special cases that \mathbf{r} lies in a chamber around the center of $C(\mathbb{D}_2^n)$. The others, though less significant for the Kähler cone of $\overline{\mathcal{M}}_{0, n}$, can be treated similarly. There will be some differences between the case when n is odd and the case when n is even.

8.2. The odd case. This is the nicer case. There is a unique chamber C_0 that contains the half ray $\mathbb{R}_+ \cdot (1, \dots, 1)$ in $C(\mathbb{D}_2^n)$. In terms of inequality, C_0 is defined by

$$\begin{aligned} 2 \sum_{j \in J} r_j &< \sum_{1 \leq j \leq n} r_j \text{ if } |J| < \frac{n}{2}; \\ 2 \sum_{j \in J} r_j &> \sum_{1 \leq j \leq n} r_j \text{ if } |J| > \frac{n}{2}. \end{aligned}$$

Equivalently, $C_0 \cap \mathbb{D}_2^n$ is defined by

$$\sum_{j \in J} r_j < 1 \text{ if } |J| < \frac{n}{2};$$

$$\sum_{j \in J} r_j > 1 \text{ if } |J| > \frac{n}{2}.$$

Recall that all $M_{\mathbf{r}}$ underly a common projective variety M_{C_0} when \mathbf{r} lies in the interior of C_0 . It is well known that the second Betti number or the Picard number of this projective variety is equal to n . Many people have made independent calculations. The following provides one of them.

Theorem 8.3. *M_{C_0} can be obtained from \mathbb{P}^{n-3} by the following sequence of “blowups” and “blowdowns”: we start with blowing up $\binom{n-1}{1}$ points (in general position) of \mathbb{P}^{n-3} to $\binom{n-1}{1} \mathbb{P}^{n-4}$; then blowing down $\binom{n-1}{2} \mathbb{P}^1$ to $\binom{n-1}{2}$ points followed by blowing up $\binom{n-1}{2}$ points to $\binom{n-1}{2} \mathbb{P}^{n-5}$; then blowing down $\binom{n-1}{3} \mathbb{P}^2$ to $\binom{n-1}{3}$ points followed by blowing up $\binom{n-1}{3}$ points to $\binom{n-1}{3} \mathbb{P}^{n-6}$; ...; finally, blowing down $\binom{n-1}{\frac{n-3}{2}} \mathbb{P}^{\frac{n-5}{2}}$ to $\binom{n-1}{\frac{n-3}{2}}$ points followed by blowing up $\binom{n-1}{\frac{n-3}{2}}$ points to $\binom{n-1}{\frac{n-3}{2}} \mathbb{P}^{\frac{n-3}{2}}$.*

Proof. (Outline.) Take a general point in a favorable chamber of \mathbb{D}_2^n . Consider the line segment joining this point and the barycentre of \mathbb{D}_2^n . By Theorem 2.2 of [11], one get a sequence of projective morphisms between \mathbb{P}^{n-3} and M_{C_0} such that the dimensions of special fibers change according to the rule

$$d + e = n - 4,$$

starting with $0 + (n - 4)$, then $1 + (n - 5)$, and so on ..., till we arrive at the center of \mathbb{D}_2^n and end up with $\frac{n-5}{2} + \frac{n-3}{2}$. Taking into account of the number of walls that we have crossed, we obtain the assertion as in the theorem. \square

The theorem immediately implies that

$$\begin{aligned} \mathbf{P}_t(M_{C_0}) &= \mathbf{P}_t(\mathbb{P}^{n-3}) + \binom{n-1}{1} [\mathbf{P}_t(\mathbb{P}^{n-4}) - 1] + \binom{n-1}{2} [\mathbf{P}_t(\mathbb{P}^{n-5}) - \mathbf{P}_t(\mathbb{P}^1)] \\ &\quad + \cdots + \binom{n-1}{\frac{n-3}{2}} [\mathbf{P}_t(\mathbb{P}^{\frac{n-3}{2}}) - \mathbf{P}_t(\mathbb{P}^{\frac{n-5}{2}})]. \end{aligned}$$

That is,

$$\begin{aligned} \mathbf{P}_t(M_{C_0}) &= \frac{t^{2(n-2)} - 1}{t^2 - 1} + \binom{n-1}{1} t^2 \frac{t^{2(n-4)} - 1}{t^2 - 1} + \binom{n-1}{2} t^4 \frac{t^{2(n-6)} - 1}{t^2 - 1} \\ &\quad + \cdots + \binom{n-1}{\frac{n-3}{2}} t^{n-3}. \end{aligned}$$

Here $\mathbf{P}_t(X)$ denotes the Poincaré polynomial of X . These Betti numbers were probably first computed by Kirwan.

Now we turn to the Kähler cone of M_{C_0} . Define a map

$$\theta_0 : C_0 \rightarrow K(M_{C_0})$$

by

$$r \rightarrow [\omega_{\mathbf{r}}],$$

where $[\omega_{\mathbf{r}}]$ is the cohomology class of $\omega_{\mathbf{r}}$. This is initially defined in the interior of C_0 but can be extended easily to the boundary.

Theorem 8.4. *Assume $n \geq 5$. The Kähler cone $K(M_{C_0})$ of M_{C_0} can be naturally identified with the cone C_0 by the linear isomorphism $\theta : C_0 \rightarrow K(M_{C_0})$.*

Proof. By Duistermaat-Heckman's theorem⁸, θ is linear. Since $\dim C_0 = \dim K(M_{C_0}) = n$, to prove that θ is a linear isomorphism, it suffices to argue that θ_0 is surjective onto the open Kähler cone. Because the Kähler cone is generated by the cone of ample divisors (in this case), it is enough to consider integral points. Let L be any ample line bundle over $M_{C_0} \cong (\mathbb{P}^1)_{ss}^n(\mathbf{r}_{C_0})//GL(2, \mathbb{C})$ where \mathbf{r}_{C_0} is any fixed integral point in the interior of C_0 . Let $\pi : (\mathbb{P}^1)_{ss}^n(\mathbf{r}_{C_0}) \rightarrow (\mathbb{P}^1)_{ss}^n(\mathbf{r}_{C_0})//GL(2, \mathbb{C})$ be the quotient map. Then π^*L extends canonically to an ample line bundle \mathcal{L} over $(\mathbb{P}^1)^n$ because the complement $(\mathbb{P}^1)^n \setminus (\mathbb{P}^1)_{ss}^n(\mathbf{r}_{C_0})$ has codimension greater than 1. Now the surjectivity follows from the fact that $(\mathbb{P}^1)_{ss}^n(\mathbf{r}_{C_0}) = (\mathbb{P}^1)_{ss}^n(\mathcal{L})$ and \mathcal{L} descends to L . It remains to show that the image $\theta(\partial C_0)$ lies on the boundary of $K(M_{C_0})$. This basically follows from the fact that when \mathbf{r} approaches the boundary of C_0 , one gets a non-trivial (symplectic) blow down map, and in particular, for any curve S that is contracted to a point the pairing $\langle [\omega_{\mathbf{r}}], [S] \rangle$ approaches zero as \mathbf{r} approaches the boundary of C_0 . Here $[S]$ is the homology class of S . \square

Remark 8.5. A few remarks are in order. First, C_0 is the unique chamber that is invariant under permutations of coordinates. Second, M_{C_0} is the only quotient that admits an induced action of the permutation group Σ_n (which acts on $(\mathbb{P}^1)^n$ by permuting the coordinates). This is so because $(\mathbb{P}^1)_{ss}^n(\mathbf{r}_{C_0})$ is the only semi-stable set that is invariant under Σ_n .

⁸ Strictly speaking, to apply Duistermaat-Heckman's theorem in this case, one should use the Grassmannian construction of $M_{\mathbf{r}}$ as mentioned in 2.3.

The even cases are somewhat complicated due to the fact that the half ray $E = \mathbb{R}_+ \cdot (1, \dots, 1)$ lies in the intersection of a number of chambers.

First we record a result on M_E .

Theorem 8.6. *M_E can be obtained from \mathbb{P}^{n-3} by the following sequence of “blowups” and “blowdowns”: we start with blowing up $\binom{n-1}{1}$ points to $\binom{n-1}{1} \mathbb{P}^{n-4}$; then blowing down $\binom{n-1}{2} \mathbb{P}^1$ to $\binom{n-1}{2}$ points followed by blowing up $\binom{n-1}{2}$ points to $\binom{n-1}{2} \mathbb{P}^{n-5}$; then blowing down $\binom{n-1}{3} \mathbb{P}^2$ to $\binom{n-1}{3}$ points followed by blowing up $\binom{n-1}{3}$ points to $\binom{n-1}{3} \mathbb{P}^{n-6}$; ...; then blowing down $\binom{n-1}{\frac{n-4}{2}} \mathbb{P}^{\frac{n-6}{2}}$ to $\binom{n-1}{\frac{n-4}{2}}$ points followed by blowing up $\binom{n-1}{\frac{n-4}{2}}$ points to $\binom{n-1}{\frac{n-4}{2}} \mathbb{P}^{\frac{n-2}{2}}$; finally, blowing down $\binom{n-1}{\frac{n-2}{2}} \mathbb{P}^{\frac{n-4}{2}}$ to $\binom{n-1}{\frac{n-2}{2}}$ points.*

Proof. The proof is similar to that of Theorem 8.3. \square

It follows that

$$\begin{aligned} \mathbf{P}_t(M_E) &= \mathbf{P}_t(\mathbb{P}^{n-3}) + \binom{n-1}{1} [\mathbf{P}_t(\mathbb{P}^{n-4}) - 1] + \binom{n-1}{2} [\mathbf{P}_t(\mathbb{P}^{n-5}) - \mathbf{P}_t(\mathbb{P}^1)] \\ &\quad + \dots + \binom{n-1}{\frac{n-4}{2}} [\mathbf{P}_t(\mathbb{P}^{\frac{n-2}{2}}) - \mathbf{P}_t(\mathbb{P}^{\frac{n-6}{2}})]. \end{aligned}$$

That is,

$$\begin{aligned} \mathbf{P}_t(M_E) &= \frac{t^{2(n-2)} - 1}{t^2 - 1} + \binom{n-1}{1} t^2 \frac{t^{2(n-4)} - 1}{t^2 - 1} + \binom{n-1}{2} t^4 \frac{t^{2(n-6)} - 1}{t^2 - 1} \\ &\quad + \dots + \binom{n-1}{\frac{n-4}{2}} t^{n-4} \frac{t^4 - 1}{t^2 - 1}. \end{aligned}$$

Here $\mathbf{P}_t(X)$ denotes the intersection Poincaré polynomial of X ([5]). These intersection Betti numbers are first computed by Kirwan.

Now, take any maximal chamber C_* that contains the half ray E (the symbol $*$ stresses on the choice of a base point to single out a chamber). Again, the Picard number of M_{C_*} is equal to n , and we can define a map

$$\theta : C_* \rightarrow K(M_{C_*}),$$

in the same way as before. A similar proof as in Theorem 8.4 will give

Theorem 8.7. *Assume $n \geq 6$. The Kähler cone $K(M_{C_*})$ of M_{C_*} can be naturally identified with the cone C_* by the linear isomorphism $\theta : C_* \rightarrow K(M_{C_0})$.*

9. THE KÄHLER CONE OF $\overline{\mathcal{M}}_{0,n}$

9.1. By the virtue of Remark 6.6, $\overline{\mathcal{M}}_{0,n}$ carries (classes of) symplectic forms that are transported from (the classes of) $\Omega_{\mathbf{r},\varepsilon}$ by the isomorphism $\gamma : \mathfrak{M}_{\mathbf{r},\varepsilon} \rightarrow \overline{\mathcal{M}}_{0,n}$. A quick computation on the number of independent parameters for $\Omega_{\mathbf{r},\varepsilon}$ leads to $n + \frac{2^n - 2 - 2n - n(n-1)}{2} = 2^{n-1} - \frac{n^2 - n + 2}{2}$ which coincides with the second Betti number or the Picard number of $\overline{\mathcal{M}}_{0,n}$. It is well-known that the Kähler cone (or dually the Mori cone of effective curves) of a projective variety is, in general, very hard to compute. Our theory, as providing a large family of (classes of) Kähler forms $\Omega_{\mathbf{r},\varepsilon}$, sheds a light on the shape of the Kähler cone of $\overline{\mathcal{M}}_{0,n}$. We shall now give some heuristic arguments and formulate a conjecture below.

9.2. To this end, let C^* be C_0 when n is odd or a choice C_* of the maximal chambers that contains the ray $\mathbb{R}_+ \cdot (1, \dots, 1)$ when n is even, and introduce a large cone built on

$$\mathbf{r} \in C^* \text{ and } \varepsilon = (\varepsilon_J)_{J \in \mathcal{R}_{>2}(\mathbf{r})} \in \mathbb{R}_+^{2^{n-1}-1-n-\frac{n(n-1)}{2}}$$

in the legal range as follows:

$$\mathcal{C} = \{(\mathbf{r}, \varepsilon) \in C^* \times \mathbb{R}_+^{2^{n-1}-1-n-\frac{n(n-1)}{2}} \mid 0 < \varepsilon_J < 2 \min\{r_j\}_{j \in J}, J \in \mathcal{R}_{>2}(\mathbf{r})\}.$$

(One can check that this is indeed a positive convex cone.) Assume that we have obtained a Kähler form $\Omega_{\mathbf{r},\varepsilon}$ for suitable ε , provided that \mathbf{r} is away from walls. This leads to a well-defined map Θ^0 from an open subcone \mathcal{C}^0 of \mathcal{C} to the Kähler cone $K(\overline{\mathcal{M}}_{0,n})$ of $\overline{\mathcal{M}}_{0,n}$ by taking the cohomology classes of $\Omega_{\mathbf{r},\varepsilon}$. One checks that $\Omega_{\mathbf{r},\varepsilon}$ depends on $(\mathbf{r}, \varepsilon)$ continuously and Θ^0 extends to a continuous map

$$\Theta : \mathcal{C}^0 \rightarrow K(\overline{\mathcal{M}}_{0,n}).$$

To exclude triviality, we assume that $n \geq 5$.

Problem 9.3. Assume that $n \geq 5$. Describe \mathcal{C}^0 and prove that the map Θ identifies the cone \mathcal{C}^0 with the Kähler cone $K(\overline{\mathcal{M}}_{0,n})$ of $\overline{\mathcal{M}}_{0,n}$.

Dually, we may also consider the Mori cone of effective curves (see a conjecture by Fulton as formulated in [16]). It should be instructive to study the two approaches all together.

We end our exposition by a digressive remark.

Remark 9.4. By 3.3.21 of [2], $C(\mathbb{D}_2^n)$ is the G -ample cone for both the $\mathrm{PGL}(2)$ -action on $(\mathbb{P}^1)^n$ and the maximal torus action on the Grassmannian $G(2, \mathbb{C}^n)$. It is known that the Chow quotients of these two

actions can be identified with $\overline{\mathcal{M}}_{0,n}$. Thus the above conjecture would establish an interesting connection between the G -ample cone of a projective G -variety and the ample cone of its Chow quotient. This and moreover the case for a general algebraic group action call for further investigation.

In a forthcoming paper, we will return to these topics.

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